Online Appendix to "DSGE Model Forecasting: Rational Expectations vs. Adaptive Learning"

Anders Warne*

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A. Learning in DSGE Models

The structural form of a DSGE model is given by:

$$H_{-1}z_{t-1} + H_0 z_t + H_1 E_t\big[z_{t+1}\big] = D\eta_t, \tag{A.1}$$

where z_t is a p-dimensional vector of model or state variables and η_t is a q-dimensional vector of i.i.d. Gaussian structural shocks with zero mean and identify covariance matrix. Under the assumption of *rational expectations* (RE), the solution to this stochastic difference equation is given by

$$z_t = F z_{t-1} + B\eta_t, \tag{A.2}$$

where F and B satisfies $B = (H_0 + H_1 F)^{-1}D$ and $H_{-1} + H_0 F + H_1 F^2 = 0$. This corresponds to the state equation, while the measurement equation is given by

$$y_t = A'x_t + H_t' z_t + w_t, \tag{A.3}$$

where y_t is an n-dimensional vector of observable variables, x_t a k-dimensional vector of deterministic variables, while w_t is an n-dimensional i.i.d. Gaussian measurement error vector, independent of η_t and with mean zero and covariance matrix R.

Over the last decades, alternative approaches to modelling expectations have been suggested in the literature. These include but are not limited to the bounded rationality model of Sargent (1993), rational inattention as in Sims (2003), the sticky information model of Mankiw and Reis (2002), partial information as in Svensson and Woodford (2003) and the learning approach of Evans and Honkapohja (2001). Below we discuss the adaptive learning approach suggested by Slobodyan and Wouters (2012) to DSGE models.

To relax the strict implications of the RE assumption, Slobodyan and Wouters (2012) assume that agents forecast the forward looking variables of a model as a reduced form of the lagged state variables. A special case of this is given by the expression in equation (A.2), but it is also possible

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*Forecasting and Policy Modelling Division of DG-Economics, European Central Bank, 60640 Frankfurt am Main, Germany; e-mail: anders.warne@ecb.europa.eu; Phone: +49-69 1344 8737.

that the reduced form model differs from the RE solution. First, the parameters of the reduced form need not satisfy the cross equation restrictions of the RE solution. Second, the reduced form may involve additional lags of the state variables or include other variables, such as the deterministic variables x_t, which influence the law of motion of the forward looking variables in z_t. In the next section we shall first consider a transformation of the structural form where only the forward looking variables appear in the expectations term. After we have established such a rewrite of the model, the so called *perceived law of motion* (PLM) and updating of its belief parameters through a Kalman filter are considered.

A.1. A Transformation of the Structural Form

The forward looking variables in the model can be extracted from z_t in equation (A.1) by constructing a 0-1 selection matrix S of dimension $p \times f$ and having rank $f \leq p$ such that

$$z_t^f = S' z_t.$$

Note that S is made up of f distinct columns of I_p. The remaining $p - f$ columns are denoted by $S_\perp$ such that the non-forward looking variables are given by

$$z_t^{nf} = S_\perp' z_t.$$

We can now define the matrix $\tilde{S}$, which we will employed to transform (re-order) the structural equations and the state variables, as follows:

$$\tilde{S} = \begin{bmatrix} S & S_\perp \end{bmatrix},$$

where $\tilde{S}^{-1} = \tilde{S}'$.

The order of the equations in the DSGE model is arbitary and for the transformation approach below it is important that the order at least temporarily follows the order in which the forward looking variables appear in z. Moreover and assuming expectations of exogenous shock processes have been substituted for, it is required that

$$\mathrm{rank}\big[H_1\big] \leq f,$$

i.e., the rank of H_1 provides a lower bound for the number of forward looking variables that are supported by the model.[1]

Concerning the reordering of equations, each forward looking variable in the expectation term should appear in an equation having the same order number as the variable itself has among z. To

[1] See, for instance, equation (29) in An and Schorfheide (2007), where two exogenous and backward looking variables appear in expectation, government consumption and technology. Although a total of four variables have nonzero elements in the H_1 matrix of their formulation of the log-linearized DSGE model and the rank of H_1 is two, only two variables can be forward looking. In their fomulation of the model, the forward looking variables are output and inflation, but its also possible to rewrite the model and replace output as a forward looking variable with consumption.

this end, the matrix C is defined as a $p \times p$ matrix of rank p containing only unique rows from I_p. This means that C satisfies $C'C = CC' = I_p$.[2]

Premultiplying the system in (A.1) by $\tilde{S}'C$, it can be rewritten as follows

$$\tilde{H}_{-1}\tilde{z}_{t-1} + \tilde{H}_0\tilde{z}_t + \tilde{H}_1 E_t\bigl[\tilde{z}_{t+1}\bigr] = \tilde{D}\eta_t, \tag{A.4}$$

where $\tilde{z}_t = \tilde{S}'z_t$, $\tilde{H}_i = \tilde{S}'CH_i\tilde{S}$ for $i = -1, 0, 1$, and $\tilde{D} = \tilde{S}'CD$. This means that the equations for the forward looking variables are ordered in the top f equations (rows) and the bottom $p - f$ equations (rows) are those for the non-forward looking variables. Furthermore, the former variables are given by the first f elements of $\tilde{z}_t$ and the latter variables by the bottom $p - f$ elements.

The matrices $\tilde{H}_i$ can be expressed in matrix blocks as follows

$$\tilde{H}_i = \begin{bmatrix} S'CH_iS & S'CH_iS_\perp \\ S_\perp'CH_iS & S_\perp'CH_iS_\perp \end{bmatrix}, \quad i = -1, 0, 1.$$

In the case of $\tilde{H}_1$, the $f \times f$ sub-matrix $S'CH_1S$ has full rank f, while the sub-matrix $S_\perp'CH_1S_\perp = 0$. These results follow directly from the assumptions that z_t^f is forward looking and that z_t^{nf} is non-forward looking. For the sub-matrix in the bottom right corner of $\tilde{H}_1$ to be zero, we find that either $CH_1S_\perp = 0$ or $S_\perp'CH_1 = 0$.

For the case when $CH_1S_\perp = 0$, we find that the final $p - f$ columns of $\tilde{H}_1$ are zero and that only the expected next period values of the forward looking variables appear in the p equations. There is therefore no need for any further transformation of the DSGE model as

$$\tilde{H}_1 E_t\bigl[\tilde{z}_{t+1}\bigr] = \tilde{H}_{1,f} E_t\bigl[z_{t+1}^f\bigr].$$

Based on this we can replace the expectations with the adaptive learning mechanism for the forward looking variables when we solve the model.

The second case with $S_\perp'CH_1 = 0$ is somewhat more complicated as we need to transform the system by substituting for the expectations of the non-forward looking variables in the top f equations. To accomplish this, we note that the bottom $p - f$ equations in (A.4) do not involve any expectations, but only contemporaneous and lagged variables. Under the assumption that the model has a solution, it follows that

$$\operatorname{rank}\bigl(S_\perp'CH_0S_\perp\bigr) = p - f.$$

Accordingly, the solution of the DSGE model includes the following representation for the non-forward looking variables

$$\begin{aligned} z_t^{nf} = {} & -\bigl(S_\perp'CH_0S_\perp\bigr)^{-1}S_\perp'CH_0Sz_t^f - \bigl(S_\perp'CH_0S_\perp\bigr)^{-1}\Bigl[S_\perp'CH_{-1}S \;\; S_\perp'CH_{-1}S_\perp\Bigr]\tilde{z}_{t-1} \\ & + \bigl(S_\perp'CH_0S_\perp\bigr)^{-1}S_\perp'D\eta_t. \end{aligned}$$

[2] Let ι_f denote an f dimensional vector with integers giving the position of each forward looking variable in z. Similarly, let ι_e be an f dimensional vector with positions of the rows in H_1 that are non-zero. This vector need not be unique as more rows than f may contain non-zero elements. For such situations, it is only required that the sub-matrix formed from H_1 using the rows ι_e and the columns ι_f has rank f. Provided that this condition is met, C is constructed by first setting it to I_p. Next, the rows ι_e in C are replaced with the rows ι_f from I_p. Last, the rows ι_f of C are replaced with ι_e of I_p.

From this equation we see that the unbiased expectation of the non-forward looking variables at $t+1$ based on information at t is

$$E_t\bigl[z_{t+1}^{nf}\bigr] = -\bigl(S_\perp' C H_0 S_\perp\bigr)^{-1} S_\perp' C H_0 S E_t\bigl[z_{t+1}^{f}\bigr] - \bigl(S_\perp' C H_0 S_\perp\bigr)^{-1}\Bigl[S_\perp' C H_{-1} S \;\; S_\perp' C H_{-1} S_\perp\Bigr]\tilde{z}_t.$$

Substituting this into equation (A.4), making use of $S_\perp' C H_1 = 0$ and collecting terms, we obtain

$$\bar{H}_{-1}\tilde{z}_{t-1} + \bar{H}_0\tilde{z}_t + \bar{H}_1 E_t\bigl[\tilde{z}_{t+1}\bigr] = \tilde{D}\eta_t, \tag{A.5}$$

where $\bar{H}_{-1} = \tilde{H}_{-1}$,

$$\bar{H}_0 = \tilde{H}_0 - \begin{bmatrix} S' C H_1 S_\perp\bigl(S_\perp' C H_0 S_\perp\bigr)^{-1} S_\perp' C H_{-1} S & S' C H_1 S_\perp\bigl(S_\perp' C H_0 S_\perp\bigr)^{-1} S_\perp' C H_{-1} S_\perp \\ 0 & 0 \end{bmatrix},$$

while

$$\bar{H}_1 = \begin{bmatrix} S' C H_1 S - S' C H_1 S_\perp\bigl(S_\perp' C H_0 S_\perp\bigr)^{-1} S_\perp' C H_0 S & 0 \\ 0 & 0 \end{bmatrix}.$$

For this transformation we find that

$$\bar{H}_1 E_t\bigl[\tilde{z}_{t+1}\bigr] = \bar{H}_{1,f} E_t\bigl[z_{t+1}^{f}\bigr].$$

At this stage, it is useful to premultiply the structural form by $C'\tilde{S}$ and replace $\tilde{z}_t$ and $\tilde{z}_{t-1}$ with z_t and z_{t-1}, respectively, while the expectations term is kept. This provides us with

$$H_{-1}^{*} z_{t-1} + H_0^{*} z_t + H_{1,f}^{*} E_t\bigl[z_{t+1}^{f}\bigr] = D\eta_t. \tag{A.6}$$

The structural form matrices are now given by $H_{-1}^{*} = H_{-1}$,

$$H_0^{*} = \begin{cases} C'\tilde{S}\tilde{H}_0\tilde{S}' = H_0 & \text{if } C H_1 S_\perp = 0, \\ C'\tilde{S}\bar{H}_0\tilde{S}' & \text{if } S_\perp' C H_1 = 0, \end{cases}$$

and

$$H_{1,f}^{*} = \begin{cases} C'\tilde{S}\tilde{H}_{1,f} = H_1 S & \text{if } C H_1 S_\perp = 0, \\ C'\tilde{S}\bar{H}_{1,f} & \text{if } S_\perp' C H_1 = 0. \end{cases}$$

These conditions and transformations are simple to apply once the forward looking variables have been established. The case when all columns of H_1 that are multiplied by a non-forward looking variable are zero is very easy to spot and require hardly any rewrite of the DSGE model. The second case when all rows of H_1 in the equations for the non-forward looking variables are zero, require a little bit more work, but is swiftly dealt with by computer code.

Finally, the requirement that f is at least equal to the rank of H_1 hints at a deeper issue concerning the uniqueness of solutions under adaptive learning. Different but equivalent formulations of a DSGE model from a rational expectations perspective can all have the same unique solution, but this is not necessarily the case when rational expectations are replaced with adaptive learning. Rather, the choice of forward looking variables has a direct implication for the solution. How different the solutions are for the various possibilities is an empirical question, but it needs to be recognized. Furthermore, the matrix H_1 has rank 5 in the Smets and Wouters (2007) model studied by Slobodyan and Wouters (2012) while the number of nonzero columns of this matrix is 7. This means that we may choose between five and seven forward looking variables. At the same time, not all of the seven candidates can be selected if we set $f = 5$. For any valid selection of five forward looking variables, the matrix $H_1 S$ must have rank $f = 5$.

A.2. The Perceived Law of Motion and Kalman Filtering

When applying adaptive learning to a DSGE model, it is important to make sure that all the forward looking variables are endogenous, i.e., that some of them are *not* specified as being backward looking, such as a shock processes. This would clearly lead to an inconsistent system since we cannot have two separate backward looking equations for the same variable.

Slobodyan and Wouters (2012) assume that each forward looking variable, $z^f_{t,j}$, is determined by a limited number of variables, denoted by $q_{t-1,j}$ and having dimension g_j, through the PLM

$$z^f_{t,j} = q'_{t-1,j}\beta_{t-1,j} + u_{t,j}, \quad j = 1, \ldots, f. \tag{A.7}$$

In their benchmark case, $q_{t-1,j}$ is given by a constant and two lags of $z^f_{t,j}$. Below we consider a general expression for the PLM, with the only restriction that it can only contain z variables and deterministic variables.

Stacking the forecasting model in equation (A.7) in SURE form yields

$$z^f_t = q'_{t-1}\beta_{t-1} + u_t, \tag{A.8}$$

where u_t is i.i.d. Gaussian with zero mean and covariance matrix

$$E\bigl[u_t u'_t\bigr] = \Sigma_u,$$

an $f \times f$ positive definite matrix. Furthermore, we have that

$$q_{t-1} = \begin{bmatrix} q_{t-1,1} & 0 & \cdots & 0 \\ 0 & q_{t-1,2} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & q_{t-1,f} \end{bmatrix},$$

is a $g \times f$ matrix with $g = \sum_{j=1}^{f} g_j$. Consequently, β_t is a time-varying vector of dimension g, capturing how the PLM changes over time.

The β_t vector is estimated with a Kalman filter which treats (A.8) as a measurement equation. The state equation is assumed to be

$$\beta_t = \beta + \Gamma\bigl(\beta_{t-1} - \beta\bigr) + \varepsilon_t, \tag{A.9}$$

where β is the initial belief (steady-state) for the unobserved β_t-process. The $g \times g$ matrix Γ is assumed to be a diagonal matrix with

$$\Gamma = \rho I_g,$$

where the scalar parameter $0 \leq \rho \leq 1$. The g-dimensional vector of errors ε_t is assumed to be i.i.d. Gaussian with mean zero, independent of u_t, and with

$$E\big[\varepsilon_t \varepsilon_t'\big] = \Sigma_\varepsilon,$$

a $g \times g$ positive definite matrix.

The Kalman filter updating and forecasting equations for the belief coefficients and their covariance matrices are given by

$$\beta_{t|t} = \beta_{t|t-1} + R_{t|t-1} q_{t-1} \left(q_{t-1}' R_{t|t-1} q_{t-1} + \Sigma_u \right)^{-1} \big(z_t^f - q_{t-1}' \beta_{t-1|t-1} \big), \tag{A.10}$$

$$\beta_{t+1|t} = \beta + \Gamma \big(\beta_{t|t} - \beta \big), \tag{A.11}$$

$$R_{t|t} = R_{t|t-1} - R_{t|t-1} q_{t-1} \left(q_{t-1}' R_{t|t-1} q_{t-1} + \Sigma_u \right)^{-1} q_{t-1}' R_{t|t-1}, \tag{A.12}$$

$$R_{t+1|t} = \Gamma R_{t|t} \Gamma' + \Sigma_\varepsilon. \tag{A.13}$$

Notice that $q_{t-1}' \beta_{t-1|t-1}$ is the one-step-ahead forecast of z_t^f, while $q_{t-1}' R_{t|t-1} q_{t-1} + \Sigma_u$ is the forecast error covariance matrix.

The estimation of β_t is based on all variables in the PLM being observable. In practise, however, we observe y_t, a smaller dimensional vector such that z_t^f and q_{t-1} need to be replaced with $z_{t|t}^f$ and $q_{t-1|t-1}$, respectively. We return to the details in Section A.5.2 where we introduce the full algorithm. In addition, the parameters β, Σ_u, Σ_ε, as well as the initial values for $\beta_{1|0}$ and $R_{1|0}$ need to be determined. We turn to this problem next.

A.3. Initial Values for the Beliefs

The benchmark approach suggested by Slobodyan and Wouters (2012) is to use population moments to determine the initial values for the belief parameters. Specifically, they let

$$\beta_{1|0} = \beta = E\big[q_{t-1} q_{t-1}'; \theta\big]^{-1} E\big[q_{t-1} z_t^f; \theta\big]$$

where θ is the parameter vector of the DSGE model under RE. Similarly, the covariance matrix of the expectation errors is given by

$$\Sigma_u = E\left[\big(z_t^f - q_{t-1}' \beta \big) \big(z_t^f - q_{t-1}' \beta \big)'; \theta \right].$$

Finally, they let

$$R_{1|0} = \sigma_r \Big(E\big[q_{t-1} \Sigma_u^{-1} q_{t-1}'; \theta \big] \Big)^{-1},$$

$$\Sigma_\varepsilon = \sigma_\varepsilon \Big(E\big[q_{t-1} \Sigma_u^{-1} q_{t-1}'; \theta \big] \Big)^{-1},$$

where σ_r and σ_ε are positive scalars, while the $g \times g$ matrix appearing in $R_{1|0}$ and Σ_ε is called the "GLS" matrix below. The determination of parametric expressions for the population moments conditional on θ and based on the RE version of the DSGE model is the target for the discussion below. However, it may first be noted that the learning dynamics involves three unknown parameters in addition to θ. Namely, the scale parameters σ_r and σ_ε as well as the autocorrelation parameter ρ. Slobodyan and Wouters (2012) estimate the latter parameter while the scale parameters are calibrated.[3]

A.3.1. The β Vector

The population covariances for the model variables z_t based on RE can be computed from (A.2) and are given by

$$E\big[z_t z'_{t-j};\theta\big] = F^j\Sigma_\xi, \quad j = 0,1,2,\ldots, \tag{A.14}$$

where Σ_ξ satisfies $\Sigma_\xi = F\Sigma_\xi F' + BB'$. Consequently, we also have that $E[z_t^f z'_{t-j};\theta] = S'E[z_t z'_{t-j};\theta]$, $E[z_t x'_t;\theta] = 0$, while $\Sigma_x^{(0)} = (1/T)\sum_{t=1}^T x_t x'_t$. Let Z_{t-1} be a $(k+pm)$-dimensional vector such that

$$Z_{t-1} = \begin{bmatrix} x'_t & z'_{t-1} & \cdots & z'_{t-m} \end{bmatrix}',$$

while G_j is a $(k+pm)\times g_j$ selection matrix such that

$$q_{t-1,j} = G'_j Z_{t-1}, \quad j = 1,\ldots,f.$$

In other words, m is the maximum number of lags of the model variables that appear in $q_{t-1,j}$ for all f forward looking variables. The benchmark case in Slobodyan and Wouters (2012) is based on $m=2$, with G_j selecting the first and the second lag of $z_{t,j}^f$ as well as a constant from x_t for all j.

To obtain an analytical expression for β (and $\beta_{1|0}$) based on the RE version of the DSGE model, we proceed by noting that $E[q_{t-1}q'_{t-1};\theta]$ is a $g\times g$ block diagonal matrix with typical block element

$$E\big[q_{t-1,j}q'_{t-1,j};\theta\big] = G'_j E\big[Z_{t-1}Z'_{t-1};\theta\big]G_j = G'_j\Sigma_Z G_j, \quad j = 1,\ldots,f.$$

The $(k+pm)\times(k+pm)$ matrix Σ_Z is obtained from equation (A.14) as well as from the properties with the deterministic variables such that

$$\Sigma_Z = \begin{bmatrix} \Sigma_x^{(0)} & 0 & 0 & \cdots & 0 \\ 0 & \Sigma_\xi & F\Sigma_\xi & \cdots & F^{m-1}\Sigma_\xi \\ 0 & \Sigma_\xi F' & \Sigma_\xi & \cdots & F^{m-2}\Sigma_\xi \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & \Sigma_\xi(F')^{m-1} & \Sigma_\xi(F')^{m-2} & & \Sigma_\xi \end{bmatrix}.$$

This matrix is positive semidefinite, but need not be positive definite. For example, z_t may include a contemporaneous entry for a variable and the first lag of the same variable. For this case, z_t and z_{t-1} share a variable and the covariance matrix for Z_t is therefore singular when $m>1$.

[3] In their empirical application, Slobodyan and Wouters (2012) set the scale parameters to $\sigma_r = 0.03$ and $\sigma_\varepsilon = 0.003$, respectively, while ρ is provided with a standard uniform prior.

Let S_j denote column j of S such that $z_{t,j}^f = S_j' z_t^f$, Furthermore, let J be a $(k+pm) \times p$ matrix such that

$$J' = \begin{bmatrix} 0_{p\times k} & I_p & 0_{p\times p(m-1)} \end{bmatrix}.$$

It can now be shown that

$$E\big[q_{t-1} z_t^f; \theta\big] = \begin{bmatrix} G_1'\Sigma_Z JF'S_1 \\ \vdots \\ G_f'\Sigma_Z JF'S_f \end{bmatrix}.$$

It therefore follows that

$$\beta = \begin{bmatrix} \left(G_1'\Sigma_Z G_1\right)^{-1} G_1'\Sigma_Z JF'S_1 \\ \vdots \\ \left(G_f'\Sigma_Z G_f\right)^{-1} G_f'\Sigma_Z JF'S_f \end{bmatrix}, \tag{A.15}$$

a vector of equation-by-equation estimates of β_j for $j = 1, \ldots, f$.

A.3.2. The Σ_u Matrix

The derive an analytical expression of Σ_u, note that

$$q_{t-1}'\beta = \big(\beta' \otimes I_f\big)\mathrm{vec}\big(q_{t-1}'\big).$$

Since q_{t-1} is a $g \times f$ matrix, we know from Magnus and Neudecker (1988) that

$$\mathrm{vec}\big(q_{t-1}'\big) = K_{gf}\mathrm{vec}\big(q_{t-1}\big),$$

where K_{gf} is a $gf \times gf$ commutation matrix. Hence,

$$q_{t-1}'\beta = \big(\beta' \otimes I_f\big)K_{gf}\mathrm{vec}\big(q_{t-1}\big).$$

Next, define the $g \times f(k+pm)$ matrix G such that

$$G = \begin{bmatrix} G_1' & 0 & \cdots & 0 \\ 0 & G_2' & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & G_f' \end{bmatrix} = \begin{bmatrix} \tilde{G}_1 & \cdots & \tilde{G}_f \end{bmatrix},$$

where $\tilde{G}_j$ is $g \times (k+pm)$ containing columns $(k+pm)(j-1)+1$ until $(k+pm)j$ of G for $j = 1, \ldots, f$. Furthemore, let $\tilde{G}$ be a $gf \times (k+pm)$ matrix with

$$\tilde{G} = \begin{bmatrix} \tilde{G}_1 \\ \vdots \\ \tilde{G}_f \end{bmatrix}.$$

It then holds that

$$\mathrm{vec}\big(q_{t-1}\big) = \tilde{G}Z_{t-1}.$$

Accordingly,

$$q_{t-1}'\beta = \big(\beta' \otimes I_f\big) K_{gf}\tilde{G}Z_{t-1} = \tilde{\beta}Z_{t-1}. \tag{A.16}$$

With this result in mind it follows that

$$E\Big[z_t^f z_t^{f\prime};\theta\Big] = S'\Sigma_\xi S = S'J'\Sigma_Z JS, \tag{A.17}$$

$$E\Big[q_{t-1}'\beta z_t^{f\prime};\theta\Big] = \tilde{\beta}\Sigma_Z JF'S, \tag{A.18}$$

$$E\Big[q_{t-1}'\beta\beta' q_{t-1};\theta\Big] = \tilde{\beta}\Sigma_Z\tilde{\beta}'. \tag{A.19}$$

Accordingly, an analytical expression of Σ_u is obtained from

$$\Sigma_u = S'J'\Sigma_Z JS - \tilde{\beta}\Sigma_Z JF'S - S'FJ'\Sigma_Z\tilde{\beta}' + \tilde{\beta}\Sigma_Z\tilde{\beta}'. \tag{A.20}$$

A.3.3. The GLS Matrix

Let the inverse of the Σ_u matrix be expressed as

$$\Sigma_u^{-1} = \begin{bmatrix} \omega_{11} & \cdots & \omega_{1f} \\ \vdots & & \vdots \\ \omega_{1f} & \cdots & \omega_{ff}. \end{bmatrix}$$

It is then straightforward to show that the $g \times g$ inverse of the GLS matrix is given by:

$$E\big[q_{t-1}\Sigma_u^{-1}q_{t-1}';\theta\big] = \begin{bmatrix} \omega_{11}G_1'\Sigma_Z G_1 & \cdots & \omega_{1f}G_1'\Sigma_Z G_f \\ \vdots & & \vdots \\ \omega_{1f}G_f'\Sigma_Z G_1 & \cdots & \omega_{ff}G_f'\Sigma_Z G_f \end{bmatrix}. \tag{A.21}$$

A.4. The Actual Law Of Motion

To determine the actual law of motion (ALM) of all variables in z_t we first solve for the expectations in equation (A.1). The derivations employed to obtain equation (A.16) can be used to show that

$$q_t'\beta_t = \tilde{\beta}_t Z_t,$$

where

$$\tilde{\beta}_t = \big(\beta_t' \otimes I_f\big) K_{gf}\tilde{G}.$$

This means that

$$E_t\big[z_{t+1}^f\big] = \tilde{\beta}_{t|t}Z_t, \tag{A.22}$$

where $\tilde{\beta}_{t|t}$ is determined in Section A.2. It is useful to decompose $\tilde{\beta}_t$ such that

$$\tilde{\beta}_t = \Big[\tilde{\beta}_{t,0} \quad \tilde{\beta}_{t,1} \quad \cdots \quad \tilde{\beta}_{t,m}\Big], \tag{A.23}$$

where $\tilde{\beta}_{t,0}$ is an $f \times k$ matrix and $\tilde{\beta}_{t,i}$ is an $f \times p$ matrix for $i = 1, \ldots, m$.

Before we turn to the general solution, let us assume that $m = 2$ and $x_t = 1$ as in Slobodyan and Wouters (2012). This means that

$$E_t\left[z_{t+1}^f\right] = \tilde{\beta}_{t|t,0} + \tilde{\beta}_{t|t,1} z_t + \tilde{\beta}_{t|t,2} z_{t-1}.$$

Making use of (A.6) and the above expression for the projected forward looking variables, provides us with the following structural form of the DSGE model

$$\left[H_0^* + H_{1,f}^* \tilde{\beta}_{t|t,1}\right] z_t + \left[H_{-1}^* + H_{1,f}^* \tilde{\beta}_{t|t,2}\right] z_{t-1} + H_{1,f}^* \tilde{\beta}_{t|t,0} = D\eta_t. \tag{A.24}$$

It now follows that the ALM is given by

$$z_t = \tilde{\mu}_t + \tilde{F}_{t,1} z_{t-1} + \tilde{B}_{t,0} \eta_t, \tag{A.25}$$

where

$$\begin{aligned}
\tilde{\mu}_t &= -\left[H_0^* + H_{1,f}^* \tilde{\beta}_{t|t,1}\right]^{-1} H_{1,f}^* \tilde{\beta}_{t|t,0}, \\
\tilde{F}_{t,1} &= -\left[H_0^* + H_{1,f}^* \tilde{\beta}_{t|t,1}\right]^{-1} \left[H_{-1}^* + H_{1,f}^* \tilde{\beta}_{t|t,2}\right], \\
\tilde{B}_{t,0} &= \left[H_0^* + H_{1,f}^* \tilde{\beta}_{t|t,1}\right]^{-1} D.
\end{aligned}$$

The condition for a unique solution at t is therefore that the matrix premultiplied by z_t in equation (A.24) is invertible. The ALM in (A.25) with $\xi_t = z_t$ is now the state equation for the DSGE model with adaptive learning and it can be used together with the measurement equation in (A.3) to form the state-space representation. It should also be noted that premultiplying z_t in (A.25) with S' gives the ALM for the forward looking variables z_t^f and the corresponding equation differs from the PLM in equation (A.8). In other words, expectations are typically *not* model consistent.[4]

For the general case of a finite m and deterministic variables x_t, substitution of the PLM into the structural form and rearranging terms gives us

$$\begin{aligned}
&\left[H_0^* + H_{1,f}^* \tilde{\beta}_{t|t,1}\right] z_t + \left[H_{-1}^* + H_{1,f}^* \tilde{\beta}_{t|t,2}\right] z_{t-1} \\
&\qquad + H_{1,f}^* \sum_{j=2}^{m-1} \tilde{\beta}_{t|t,j+1} z_{t-j} + H_{1,f}^* \tilde{\beta}_{t|t,0} x_t = D\eta_t.
\end{aligned} \tag{A.26}$$

It follows that a unique solution at t exists under the same conditions as when $m = 2$ and it is given by

$$z_t = \tilde{\mu}_t x_t + \sum_{j=1}^{m^*} \tilde{F}_{t,j} z_{t-j} + \tilde{B}_{t,0} \eta_t, \tag{A.27}$$

where $m^* = \max\{m-1, 1\}$, while $\tilde{\mu}_t$ and $\tilde{B}_{t,0}$ are given by the expressions below equation (A.25), while

$$\tilde{F}_{t,j} = \begin{cases} -\left[H_0^* + H_{1,f}^* \tilde{\beta}_{t|t,1}\right]^{-1} \left[H_{-1}^* + H_{1,f}^* \tilde{\beta}_{t|t,2}\right], & \text{if } j = 1, \\ -\left[H_0^* + H_{1,f}^* \tilde{\beta}_{t|t,1}\right]^{-1} H_{1,f}^* \tilde{\beta}_{t|t,j+1}, & \text{if } j = 2, \ldots, m-1. \end{cases}$$

[4] In principle, it is possible that expectations are model consistent when $\beta_t = \beta$ for all t and when $q_{t-1,j}$ has been selected to exactly match the variables that appear in the RE solution for all forward looking variables.

Notice that $m = 1$ implies that $\tilde{\beta}_{t|t,2} = 0$ for all t since z_{t-1} no longer appears in Z_t, the vector of variables in the PLM.

Finally, it should be noted that when $m \geq 3$, the state equation for the Kalman filter of the DSGE model is given by

$$\begin{bmatrix} z_t \\ z_{t-1} \\ z_{t-2} \\ \vdots \\ z_{t-m+2} \end{bmatrix} = \begin{bmatrix} \tilde{\mu}_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_t + \begin{bmatrix} \tilde{F}_{t,1} & \tilde{F}_{t,2} & \cdots & \tilde{F}_{t,m-2} & \tilde{F}_{t,m-1} \\ I_p & 0 & \cdots & 0 & 0 \\ 0 & I_p & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & I_p & 0 \end{bmatrix} \begin{bmatrix} z_{t-1} \\ z_{t-2} \\ z_{t-3} \\ \vdots \\ z_{t-m+1} \end{bmatrix} + \begin{bmatrix} \tilde{B}_{t,0} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \eta_t, \tag{A.28}$$

or more compactly

$$\xi_t = \tilde{M}_t x_t + \tilde{F}_t \xi_{t-1} + \tilde{B}_t \eta_t. \tag{A.29}$$

The vector ξ_t is r-dimensional with $r = pm^*$. In case $m = 1$, a unique solution is given by equation (A.25) with $\tilde{\beta}_{t|t,2} = 0$ in the expression for $\tilde{F}_{t,1}$. The general expression in (A.29) is employed for all m cases below.

A.5. A Kalman Filter for the DSGE Model with Adaptive Learning

Before we discuss the filtering and updating equations, it is useful to take a step back and consider which information is available at time t and how that can be used. First, the Kalman filter for the belief coefficients in Section A.2 is based on having observed z_t^f at t or having an update estimate $z_{t|t}^f$ based on y_t. With this information, the filter and update equations can be executed, yielding $\beta_{t|t}$ which can then be used to form the solution matrices $\tilde{M}_t$, $\tilde{F}_t$ and $\tilde{B}_t$, representing the state equation for the DSGE model under adaptive learning in (A.29). But to run the Kalman filter using this state equation requires that the values for these matrices are known at $t-1$, which is not the case.

There are now two options: Since the only unknown component of the state equation matrices is $\beta_{t|t}$, we may either replace it with $\beta_{t|t-1}$ or with $\beta_{t-1|t-1}$. Formally, the first option seems like a natural choice since the belief coefficients in period t should be used when forming the expectations for period $t+1$ in the structural form. However, $\beta_{t-1|t-1}$ will be used below since the standard treatment in the learning literature is to assume that beliefs formed today are taken as given and that agents do not take into account that they will update their beliefs in the future. This is indeed the approach taken by Slobodyan and Wouters (2012).

The one-step-ahead forecast of y_t is given by

$$y_{t|t-1} = A' x_t + H_t' \xi_{t|t-1}, \tag{A.30}$$

while the covariance matrix of the observed variable forecast is

$$\Sigma_{y,t|t-1} = H_t' P_{t|t-1} H_t + R. \tag{A.31}$$

The update equation for the state variables is

$$\xi_{t|t} = \xi_{t|t-1} + P_{t|t-1} H_t \Sigma_{y,t|t-1}^{-1} \big(y_t - y_{t|t-1}\big), \tag{A.32}$$

while the update covariance matrix is

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} H_t \Sigma_{y,t|t-1}^{-1} H_t' P_{t|t-1}. \tag{A.33}$$

Finally, the state variables are projected forward

$$\xi_{t+1|t} = \tilde{M}_{t-1} x_{t+1} + \tilde{F}_{t-1} \xi_{t|t}. \tag{A.34}$$

The covariance matrix of the state variable forecast is here given by

$$P_{t+1|t} = \tilde{F}_{t-1} P_{t|t} \tilde{F}_{t-1}' + \tilde{B}_{t-1} \tilde{B}_{t-1}'. \tag{A.35}$$

Notice that the solution matrices from $t-1$ are used at t when projecting the state variables forward.

A.5.1. Initial Values for the Filter

To initialize this filter we first note that the matrices $\tilde{M}_{t-1}$, $\tilde{F}_{t-1}$ and $\tilde{B}_{t-1}$ are time-varing only due to $\beta_{t-1|t-1}$. We may therefore consider replacing the Kalman filter estimate from the belief equation with the population moment based on the RE version of the DSGE model, i.e., β in equation (A.15). It follows that

$$\begin{aligned}
\tilde{\mu} &= -\Big[H_0^* + H_{1,f}^* \tilde{\beta}_1\Big]^{-1} \tilde{\beta}_0, \\
\tilde{B}^* &= \Big[H_0^* + H_{1,f}^* \tilde{\beta}_1\Big]^{-1} D, \\
\tilde{F}_j &= \begin{cases} -\Big[H_0^* + H_{1,f}^* \tilde{\beta}_1\Big]^{-1} \Big[H_{-1}^* + H_{1,f}^* \tilde{\beta}_2\Big], & \text{if } j = 1, \\ -\Big[H_0^* + H_{1,f}^* \tilde{\beta}_1\Big]^{-1} H_{1,f}^* \tilde{\beta}_{j+1}, & \text{if } j = 2, \ldots, m-1. \end{cases}
\end{aligned}$$

where $\tilde{\beta}_i$ is obtained from a decomposition of $\tilde{\beta}$ analogous to the one of $\tilde{\beta}_t$ in (A.23). From these matrices we can initialize the vector of state variables $\xi_{1|0} = \mu_\xi$ with

$$\mu_\xi = \big(I_r - \tilde{F}\big)^{-1} \tilde{M} \bar{x},$$

where $\bar{x}$ is given by a unit element for the constant and, for instance, zeros for non-constant deterministic variables. In case $x_t = 1$, the vector $\tilde{\mu} = 0$ with the effect that $\mu_\xi = 0$. Similarly, we may let $P_{1|0}$ be initialized through the steady-state covariance matrix conditional on the parameters satisfying the Lyapunov equation:

$$\Sigma_\xi = \tilde{F} \Sigma_\xi \tilde{F}' + \tilde{B} \tilde{B}',$$

where $\tilde{B}$ is constructed from $\tilde{B}^*$ as in equation (A.28). The Lyapunov equation can, technically, be solved by vectorization, but in practise it is usually better to rely on a fast numerical method, such as the doubling algorithm; see, e.g., Warne (2023, Section 5.3).

A.5.2. A Joint Kalman Filter Algorithm for Computing the State Variables and Belief Coefficients

The two Kalman filters for β_t and ξ_t can be combined with the solution method in a straightforward manner. As initial values the algorithm requires β, Σ_u, Σ_ϵ, $\beta_{1|0}$, $R_{1|0}$, $\tilde{M}$, $\tilde{F}$, $\tilde{B}$, μ_ξ, Σ_ξ, $\xi_{1|0}$ and $P_{1|0}$, whose determination has been discussed above. We also let $\tilde{M}_0 = \tilde{M}$, $\tilde{F}_0 = \tilde{F}$ and $\tilde{B}_0 = \tilde{B}$. The algorithm runs forward over iterations $t = 1, \ldots, T$:

(1) Compute $y_{t|t-1}$ and $\Sigma_{y,t|t-1}$ using equations (A.30)–(A.31). The log-likelihood function for period t is given by

$$\log p\big(y_t \big| x_t, \mathcal{Y}_{t-1}; \theta\big) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln\big|\Sigma_{y,t|t-1}\big| + \\ -\frac{1}{2}\big(y_t - y_{t|t-1}\big)'\Sigma_{y,t|t-1}^{-1}\big(y_t - y_{t|t-1}\big).$$

(2) Compute $\xi_{t|t}$ and $P_{t|t}$ from (A.32) and (A.33);

(3) Compute $\xi_{t+1|t}$, $P_{t+1|t}$ from (A.34) and (A.35) if $t < T$;

(4) Compute the belief coefficients and the covariance matrices using equations (A.10)–(A.13), with $z_{t|t}^{f} = S'\xi_{t|t}$ from iteration t and $z_{t-j|t-1}$, $j = 1, \ldots, m$, from iteration $t-1$; if $t \leq m$, then $z_{t-j|t-1} = 0$ for $j \geq t$. The new solution matrices $\tilde{M}_t$, $\tilde{F}_t$ and $\tilde{B}_t$ are thereafter determined using $\beta_{t|t}$.

Notice that the algorithm requires smoothing whenever $m \geq 2$ since $Z_{t-1|t-1}$ is required by the Kalman filter part of the belief equation. Smooth estimates are also more generally of interest when estimating the structural shocks and the state variables using future information, such as for the full sample.

Instead of using the smooth estimate of, say, $z_{t-2|t-1}$ in $Z_{t-1|t-1}$ one may instead use the update estimate $z_{t-2|t-2}$. From email discussions with Raf Wouters, the latter possibility is used in Slobodyan and Wouters (2012).

Finally, the Kalman filter of the belief coefficients when $t \leq m$ is based on setting $z_{0|0} = z_{-1|0} = z_{0|1} = 0$, the steady state value, while deterministic variables are included with their time t values.

A.5.3. Smooth Estimates of the State Variables

The Kalman smoother algorithm presented in, for instance, Durbin and Koopman (2012) may be employed for the state variables. Specifically,

$$\xi_{t|T} = \xi_{t|t-1} + P_{t|t-1} r_{t|T}, \tag{A.36}$$

where

$$r_{t|T} = H_t \Sigma_{y,t|t-1}^{-1}\big(y_t - y_{t|t-1}\big) + \big(\tilde{F}_{t-1} - \tilde{K}_{t-1}H_t'\big)' r_{t+1|T}, \tag{A.37}$$

with the initial condition $r_{T+1|T} = 0$, and where the Kalman gain matrix is

$$\tilde{K}_{t-1} = \tilde{F}_{t-1} P_{t|t-1} H_t \Sigma_{y,t|t-1}^{-1}. \tag{A.38}$$

Furthermore, the smoothed state covariance matrix is again

$$P_{t|T} = P_{t|t-1} - P_{t|t-1} N_{t|T} P_{t|t-1}, \tag{A.39}$$

where

$$N_{t|T} = H_t \Sigma_{y,t|t-1}^{-1} H_t' + \big(\tilde{F}_{t-1} - \tilde{K}_{t-1} H_t'\big)' N_{t+1|T} \big(\tilde{F}_{t-1} - \tilde{K}_{t-1} H_t'\big), \tag{A.40}$$

and with the initial condition $N_{T+1|T} = 0$. Mote that we have chosen to compute the smoother based on the solution matrices which make use of $\beta_{t-1|t-1}$.

Returning to Step (4) of the algorithm in Section A.5.2, equations (A.36)–(A.37) apply to any $t \leq T$ so that

$$\xi_{t-j|t-1} = \xi_{t-j|t-j-1} + P_{t-j|t-j-1} r_{t-j|t-1}, \quad j = 1, \ldots, m,$$

where

$$r_{t-j|t-1} = H_{t-j} \Sigma_{y,t-j|t-j-1}^{-1} \big(y_{t-j} - y_{t-j|t-j-1}\big) + \big(\tilde{F}_{t-j-1} - \tilde{K}_{t-j-1} H_{t-j}\big)' r_{t-j+1|t-1},$$

where $r_{t|t-1} = 0$.

A.5.4. Update and Smooth Estimates of the Measurement Errors and the Structural Shocks

The update and the smooth estimates of the measurement errors and the structural shocks can be derived using equations (A.32)–(A.38). To begin with the update estimate of the measurement error is

$$w_{t|t} = R \Sigma_{y,t|t-1}^{-1} \big(y_t - y_{t|t-1}\big), \tag{A.41}$$

with covariance matrix

$$E\big[w_{t|t} w_{t|t}'\big] = R \Sigma_{y,t|t-1}^{-1} R. \tag{A.42}$$

Similarly, the update estimate of the structural shocks is

$$\eta_{t|t} = \tilde{B}_{t-2}' H_t \Sigma_{y,t|t-1}^{-1} \big(y_t - y_{t|t-1}\big). \tag{A.43}$$

The timing of the solution matrix $\tilde{B}_{t-2}$ stems from the observation that $\xi_{t|t-1}$ is computed from the solutions obtained at $t-2$ in the algorithm and employed for updating and filtering at $t-1$. The covariance matrix is therefore given by

$$E\big[\eta_{t|t} \eta_{t|t}'\big] = \tilde{B}_{t-2}' H_t \Sigma_{y,t|t-1}^{-1} H_t' \tilde{B}_{t-2}. \tag{A.44}$$

Next, the smooth estimate of the measurement error is

$$w_{t|T} = y_t - A' x_t - H_t' \xi_{t|T} = R \Sigma_{y,t|t-1}^{-1} \big(y_t - y_{t|t-1} - H_t' P_{t|t-1} \tilde{F}_{t-1}' r_{t+1|T}\big), \tag{A.45}$$

with covariance matrix

$$E\big[w_{t|T} w_{t|T}'\big] = R \Big(\Sigma_{y,t|t-1}^{-1} + \tilde{K}_{t-1}' N_{t+1|T} \tilde{K}_{t-1}\Big) R. \tag{A.46}$$

The smooth estimate of the structural shocks is here

$$\eta_{t|T} = \tilde{B}'_{t-2} r_{t|T}, \tag{A.47}$$

while its covariance matrix is

$$E\left[\eta_{t|T}\eta'_{t|T}\right] = \tilde{B}'_{t-2} N_{t|T} \tilde{B}_{t-2}. \tag{A.48}$$

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B. Additional Tables

Table B.1: Linking the vintages of the RTD to the AWM updates.

RTD vintages	RTD common start date	AWM update	AWM start date	AWM end date	RTD euro area concept	AWM euro area concept
2001Q1–2001Q4	1994Q1	2	1970Q1	1999Q4	12	12
2002Q1–2003Q2	1994Q1	3	1970Q1	2000Q4	12	12
2003Q3–2004Q2	1994Q1	4	1970Q1	2002Q4	12	12
2004Q3–2005Q3	1994Q1	5	1970Q1	2003Q4	12	12
2005Q4–2006Q2	1995Q1	5	1970Q1	2003Q4	12	12
2006Q3–2006Q4	1995Q1	6	1970Q1	2005Q4	12	12
2007Q1–2007Q2	1996Q1	6	1970Q1	2005Q4	12,13	12
2007Q3	1996Q1	7	1970Q1	2006Q4	12,13	13
2007Q4–2008Q1	1996Q1	7	1970Q1	2006Q4	13	13
2008Q2	1995Q1	7	1970Q1	2006Q4	13,15	13
2008Q3–2008Q4	1995Q1	8	1970Q1	2007Q4	15	15
2009Q1–2009Q2	1995Q1	8	1970Q1	2007Q4	16	15
2009Q3–2010Q2	1995Q1	9	1970Q1	2008Q4	16	16
2010Q3–2010Q4	1995Q1	10	1970Q1	2009Q4	16	16
2011Q1	1995Q1	10	1970Q1	2009Q4	16,17	16
2011Q2	1995Q1	10	1970Q1	2009Q4	17	16
2011Q3–2012Q2	1995Q1	11	1970Q1	2010Q4	17	17
2012Q3–2013Q2	1995Q1	12	1970Q1	2011Q4	17	17
2013Q3–2013Q4	1995Q1	13	1970Q1	2012Q4	17	17
2014Q1–2014Q2	2000Q1	13	1970Q1	2012Q4	18	17
2014Q3–2014Q4	2000Q1	14	1970Q1	2013Q4	18	18
2015Q1–2015Q2	2000Q1	14	1970Q1	2013Q4	19	18
2015Q3–2016Q1	2000Q1	15	1970Q1	2014Q4	19	19
2016Q2	1998Q1	15	1970Q1	2014Q4	19	19
2016Q3–2017Q2	1998Q1	16	1970Q1	2015Q4	19	19
2017Q3–2018Q2	1998Q2	17	1970Q1	2016Q4	19	19
2018Q3–2020Q4	1998Q2	18	1970Q1	2017Q4	19	19

Notes: Data from the AWM is always taken from 1980Q1 until the quarter prior to the RTD common start date. When two RTD euro area concepts are indicated it means that some variables are based on one of them, while others are based on the other. In all cases, the lower euro area country concept concerns unit labor cost, while the higher concept is used for the aggregation of the other variables except in 2011Q1 when also the GDP deflator, total employment and the unemployment rate is based on euro area 16. Unit labor cost is the measure underlying the calculation of nominal wages as unit labor cost times real GDP divided by total employment. Unemployment has undergone gradual changes in the definition in December 2000, March and June 2002; see, e.g., European Central Bank (2001) for details.

Table B.2: The ragged edge of the euro area RTD: Vintages with missing data for the variables.

Date	y	c	i	π	e	w	r	u
Backcast	2001Q1–2001Q2	2001Q1–2001Q2	2001Q1–2001Q2	2001Q1–2003Q3	2001Q1–2017Q3	2001Q1–2017Q7	–	–
	2002Q1	2002Q1	2002Q1					
		2003Q3	2003Q3					
		2004Q3	2004Q3	2004Q3				
		2006Q1	2006Q1	2006Q1				
		2006Q3	2006Q3	2006Q3				
		2014Q3–2015Q4	2014Q3–2015Q4	2014Q3–2015Q4				
		2016Q2	2016Q2	2016Q2				
					2018Q1	2018Q1		
		2019Q1	2019Q1	2019Q1		2019Q1		
Total	3 of 76	15 of 76	15 of 76	22 of 76	68 of 76	69 of 76	0 of 76	0 of 76
Nowcast	2001Q1–2019Q4	2001Q1–2019Q4	2001Q1–2019Q4	2001Q1–2019Q4	2001Q1–2019Q4	2001Q1–2019Q4	–	2005Q1
								2005Q3–2005Q4
								2006Q3
								2008Q1
Total	76 of 76	76 of 76	76 of 76	76 of 76	76 of 76	76 of 76	0 of 76	5 of 76

Table B.3: Prior distributions for the structural parameters of the RE and AL version of the SWU model.

parameter	density	RE P_1	RE P_2	AL P_1	AL P_2
φ	N	4.0	1.5	4.0	1.5
σ_c	N	1.0[c]	–	1.0	0.25
λ	β	0.7	0.1	0.7	0.1
σ_l	N	2.0	0.75	2.0	0.75
ξ_w	β	0.5	0.05	0.5	0.05
ξ_p	β	0.5	0.1	0.5	0.1
$\imath_w$	β	0.5	0.15	0.5	0.15
$\imath_p$	β	0.5	0.15	0.5	0.15
ϕ_p	N	1.25	0.125	1.25	0.125
ψ	β	0.5	0.15	0.5	0.15
ρ_{mp}	β	0.75	0.1	0.75	0.1
r_π	N	1.5	0.25	1.5	0.25
r_y	N	0.125	0.05	0.125	0.05
$r_{\Delta y}$	N	0.125	0.05	0.125	0.05
ξ_e	β	0.5	0.15	0.5	0.15
υ	β	0.2	0.05	0.2	0.05
$\bar{\pi}$	Γ	0.625	0.1	0.625	0.1
$\bar{\beta}$	Γ	0.25	0.1	0.25	0.1
$\bar{e}$	N	0.2	0.05	0.2	0.05
$\bar{\gamma}$	N	0.3	0.05	0.3	0.05
α	N	0.3	0.05	0.3	0.05

Notes: The columns P_1 and P_2 refer to the mean and the standard deviation of the normal (N), standardized beta (β), and gamma (Γ) distributions. The superscript c means that the parameter is calibrated. The parameter ρ_{mp} is the coefficient on the lagged interest rate in the monetary plicy rule.

TABLE B.4: Prior distributions for the parameters of the shock processes of the RE and AL versions of the SWU model and the persistence parameter of the belief coefficients of the AL model.

		RE		AL	
parameter	density	P_1	P_2	P_1	P_2
ρ	β	–	–	0.25	0.1
ρ_g	β	0.5	0.2	0.5	0.2
ρ_{ga}	β	0.5	0.2	0.5	0.2
ρ_b	β	0.5	0.2	0.5	0.2
ρ_i	β	0.5	0.2	0.5	0.2
ρ_a	β	0.5	0.2	0.5	0.2
ρ_p	β	0.5	0.2	0.0^{c}	–
ρ_w	β	0.5	0.2	0.5	0.2
ρ_r	β	0.5	0.2	0.5	0.2
ρ_s	β	0.5	0.2	0.5	0.2
σ_g	U	0	5	0	5
σ_b	U	0	5	0	5
σ_i	U	0	5	0	5
σ_a	U	0	5	0	5
σ_p	U	0	5	0	5
σ_w	U	0	5	0	5
σ_r	U	0	5	0	5
σ_s	U	0	5	0	5

NOTES: The columns P_1 and P_2 refer to the mean and the standard deviation of the standardized beta distribution and the upper and lower bound of the uniform (U) distribution. The superscript c means that the parameter is calibrated.

TABLE B.5: Posterior estimates of the structural parameter of the euro area RE and AL versions of the SWU model for the sample 1985Q1–2019Q4.

	RE				AL			
	mean	mode	5%	95%	mean	mode	5%	95%
φ	5.03	4.86	3.85	6.41	7.50	7.40	6.03	9.03
σ_c	1.0^{c}	1.0^{c}	–	–	1.07	1.06	1.04	1.09
λ	0.64	0.65	0.57	0.70	0.87	0.87	0.84	0.89
σ_l	5.41	5.40	5.16	5.68	5.38	5.32	4.96	5.82
ξ_w	0.62	0.62	0.54	0.70	0.54	0.54	0.48	0.61
ξ_p	0.80	0.79	0.76	0.84	0.80	0.80	0.76	0.84
$\imath_w$	0.25	0.21	0.12	0.40	0.19	0.18	0.10	0.29
$\imath_p$	0.22	0.20	0.09	0.36	0.23	0.22	0.14	0.33
ϕ_p	1.54	1.55	1.41	1.67	1.36	1.35	1.23	1.50
ψ	0.52	0.52	0.40	0.64	0.58	0.58	0.37	0.77
ρ_{mp}	0.87	0.88	0.84	0.90	0.93	0.93	0.90	0.94
r_π	1.42	1.39	1.19	1.66	1.87	1.86	1.59	2.18
r_y	0.19	0.19	0.15	0.24	0.08	0.08	0.04	0.14
$r_{\Delta y}$	0.02	0.20	0.00	0.04	0.03	0.03	0.01	0.04
ξ_e	0.69	0.69	0.65	0.73	0.81	0.81	0.80	0.83
υ	0.16	0.16	0.08	0.25	0.04	0.03	0.02	0.06
$\bar{\pi}$	0.58	0.58	0.46	0.69	0.58	0.57	0.46	0.71
$\bar{\beta}$	0.22	0.21	0.11	0.35	0.24	0.23	0.12	0.40
$\bar{e}$	0.18	0.18	0.17	0.19	0.16	0.16	0.15	0.17
$\bar{\gamma}$	0.14	0.14	0.10	0.18	0.19	0.19	0.16	0.21
α	0.23	0.23	0.20	0.26	0.24	0.24	0.21	0.29

NOTES: The columns for each model display the mean, the mode, and the 5% and 95% quantiles, respectively, from the posterior distributions. The superscript c means that the parameter is calibrated.

TABLE B.6: Posterior estimates of the parameters of the shock processes of the euro area RE and AL versions of the SWU model and the persistence parameter of the belief coefficients of the AL model for the sample 1985Q1–2019Q4.

	RE				AL			
	mean	mode	5%	95%	mean	mode	5%	95%
ρ	–	–	–	–	0.17	0.15	0.07	0.28
ρ_g	0.99	0.99	0.99	0.99	0.98	0.99	0.98	0.99
ρ_{ga}	0.25	0.25	0.16	0.36	0.25	0.25	0.16	0.35
ρ_b	0.92	0.92	0.89	0.95	0.36	0.37	0.23	0.48
ρ_i	0.15	0.14	0.06	0.27	0.14	0.12	0.04	0.25
ρ_a	0.98	0.98	0.98	0.99	0.93	0.94	0.91	0.95
ρ_p	0.19	0.15	0.05	0.35	0.0^{c}	0.0^{c}	–	–
ρ_w	0.73	0.77	0.56	0.86	0.85	0.85	0.80	0.89
ρ_r	0.29	0.29	0.18	0.41	0.33	0.32	0.22	0.44
ρ_s	0.98	0.98	0.97	0.99	0.98	0.98	0.97	0.99
σ_g	0.31	0.30	0.28	0.34	0.30	0.30	0.27	0.34
σ_b	0.05	0.05	0.04	0.06	0.11	0.11	0.08	0.14
σ_i	0.55	0.55	0.48	0.63	0.26	0.25	0.20	0.33
σ_a	0.49	0.48	0.41	0.60	0.54	0.52	0.44	0.66
σ_p	0.24	0.21	0.14	0.38	0.04	0.03	0.03	0.05
σ_w	0.12	0.09	0.06	0.23	0.06	0.06	0.05	0.07
σ_r	0.10	0.10	0.09	0.11	0.09	0.09	0.08	0.10
σ_s	1.02	1.01	0.92	1.14	0.82	0.81	0.72	0.93

NOTES: See Table B.5 for details.

TABLE B.7: Kolmogorov-Smirnov tests for equality of distributions of the log predictive likelihood based on the Monte Carlo Integration estimator and the normal approximation along with p-values for the sample 2001Q1–2019Q4.

	Δy		π		$\Delta y, \pi$	
h	RE	AL	RE	AL	RE	AL
0	0.40	0.32	0.40	0.16	0.40	0.24
	1.00	1.00	1.00	1.00	1.00	1.00
1	0.49	0.33	0.33	0.50	0.49	0.41
	0.97	1.00	1.00	0.97	0.97	1.00
2	0.33	0.33	0.74	0.25	0.66	0.33
	1.00	1.00	0.64	1.00	0.78	1.00
3	0.41	0.41	0.50	0.41	0.41	0.25
	1.00	1.00	0.97	1.00	1.00	1.00
4	0.42	0.42	0.42	0.41	0.50	0.42
	1.00	1.00	1.00	1.00	0.96	1.00
5	0.67	0.42	0.59	0.42	0.50	0.34
	0.76	0.99	0.88	0.99	0.96	1.00
6	0.59	0.42	0.34	0.42	0.42	0.42
	0.88	0.99	1.00	0.99	0.99	0.99
7	0.68	0.51	0.34	0.43	0.42	0.51
	0.74	0.96	1.00	0.99	0.99	0.96
8	0.69	0.34	0.34	0.77	0.34	0.60
	0.73	1.00	1.00	0.59	1.00	0.86

NOTES: Real GDP growth is denoted by Δy and GDP deflator inflation by π. The Kolmogorov-Smirnov test is here computed as $\sqrt{N_h/2} \cdot \sup_{x_i} |F^1_{N_h}(x_i) - F^2_{N_h}(x_i)|$, where $F^j_{N_h}$ is the empirical cumulative distribution function of the log predictive likelihood using estimator j, while N_h is the number of predictive likelihood values of the h-quarter-ahead forecasts. The mean and the standard deviation of the Kolmogorov distribution are roughly 0.87 and 0.26, respectively. The critical values of the test statistic for sizes 10, 5 and 1 percent are about 1.22, 1.36 and 1.63, which may be calculated from $c_\alpha = \sqrt{-(1/2)\log(\alpha/2))}$, with α being the size of the test and c_α the critical value. The p-value of the test statistic is shown below the test value and has been computed using a truncation of 1,000 for an expression of its limiting distribution; see, for instance, Marsaglia, Tsang, and Wang (2003).

TABLE B.8: Continuous ranked probability scores of the remaining observed variables for the full sample 2001Q1–2019Q4.

	Real consumption growth			Real investment growth			Total employment growth		
	CRPS		cdf-values	CRPS		cdf-values	CRPS		cdf-values
h	RE	AL	DM	RE	AL	DM	RE	AL	DM
1	−24.80	−26.27	0.03	−65.07	−69.97	0.01	−10.86	−8.44	1.00
2	−25.28	−26.92	0.08	−67.94	−70.48	0.19	−12.50	−9.90	1.00
3	−23.92	−26.60	0.02	−67.67	−68.88	0.36	−13.12	−10.78	0.99
4	−22.16	−25.93	0.00	−67.10	−67.44	0.46	−12.98	−11.30	0.98
5	−20.15	−25.02	0.00	−66.03	−66.45	0.45	−12.48	−11.47	0.95
6	−18.90	−24.48	0.00	−63.97	−65.14	0.36	−11.81	−11.50	0.82
7	−17.89	−23.81	0.00	−62.25	−64.19	0.24	−11.09	−11.42	0.18
8	−17.35	−23.28	0.01	−61.50	−63.74	0.19	−10.52	−11.34	0.12

	Real wage growth			Unemployment			Short-term nominal interest rate		
	CRPS		cdf-values	CRPS		cdf-values	ES		cdf-values
h	RE	AL	DM	RE	AL	DM	RE	AL	DM
1	−21.16	−24.17	0.00	−13.84	−13.30	0.94	−16.14	−14.07	1.00
2	−22.44	−24.70	0.00	−21.00	−18.82	0.98	−30.10	−25.16	1.00
3	−22.58	−24.40	0.05	−29.03	−24.77	0.98	−42.74	−34.86	0.99
4	−21.74	−23.43	0.14	−36.66	−30.32	0.98	−54.15	−43.09	0.98
5	−20.56	−23.06	0.08	−43.44	−35.32	0.98	−65.53	−51.33	0.98
6	−19.37	−22.61	0.03	−49.34	−39.87	0.98	−76.01	−58.46	0.98
7	−18.16	−22.33	0.00	−54.26	−43.81	0.98	−85.66	−64.35	0.99
8	−16.99	−22.12	0.00	−58.12	−46.96	0.99	−94.60	−69.45	0.99

NOTES: The cdf-values from the Diebold-Mariano (DM) test for the null hypothesis of equal CRPS are calculated as in Harvey, Leybourne, and Newbold (1997), equation (9), with minus the CRPS values replacing the mean-squared forecast errors. The cdf-values shown above are taken from the Student's t-distribution with $N_h - 1$ degrees of freedom, with N_h being the number of h-quarter-ahead forecasts, $N_h = 76 - h$. A cdf-value close to zero suggests that the predictions of the RE model are better in a CRPS sense than those of the AL model, and a value close to one that the reverse case is supported.

C. Additional Figures

Figure C.1: The observed variables from the full sample, 1985Q1–2019Q4.

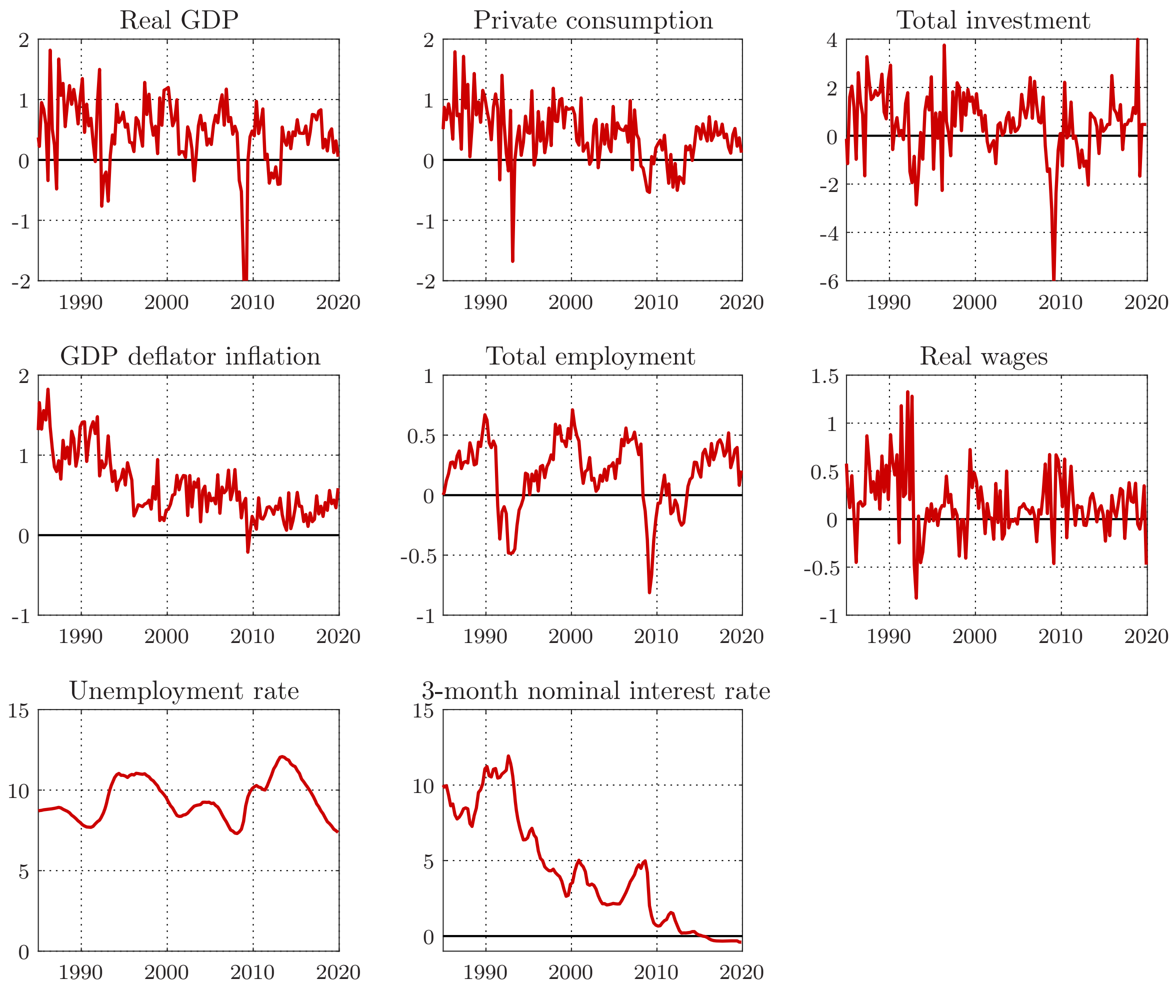


FIGURE C.2: Prediction errors of real GDP growth covering the vintages 2001Q1–2019Q4.

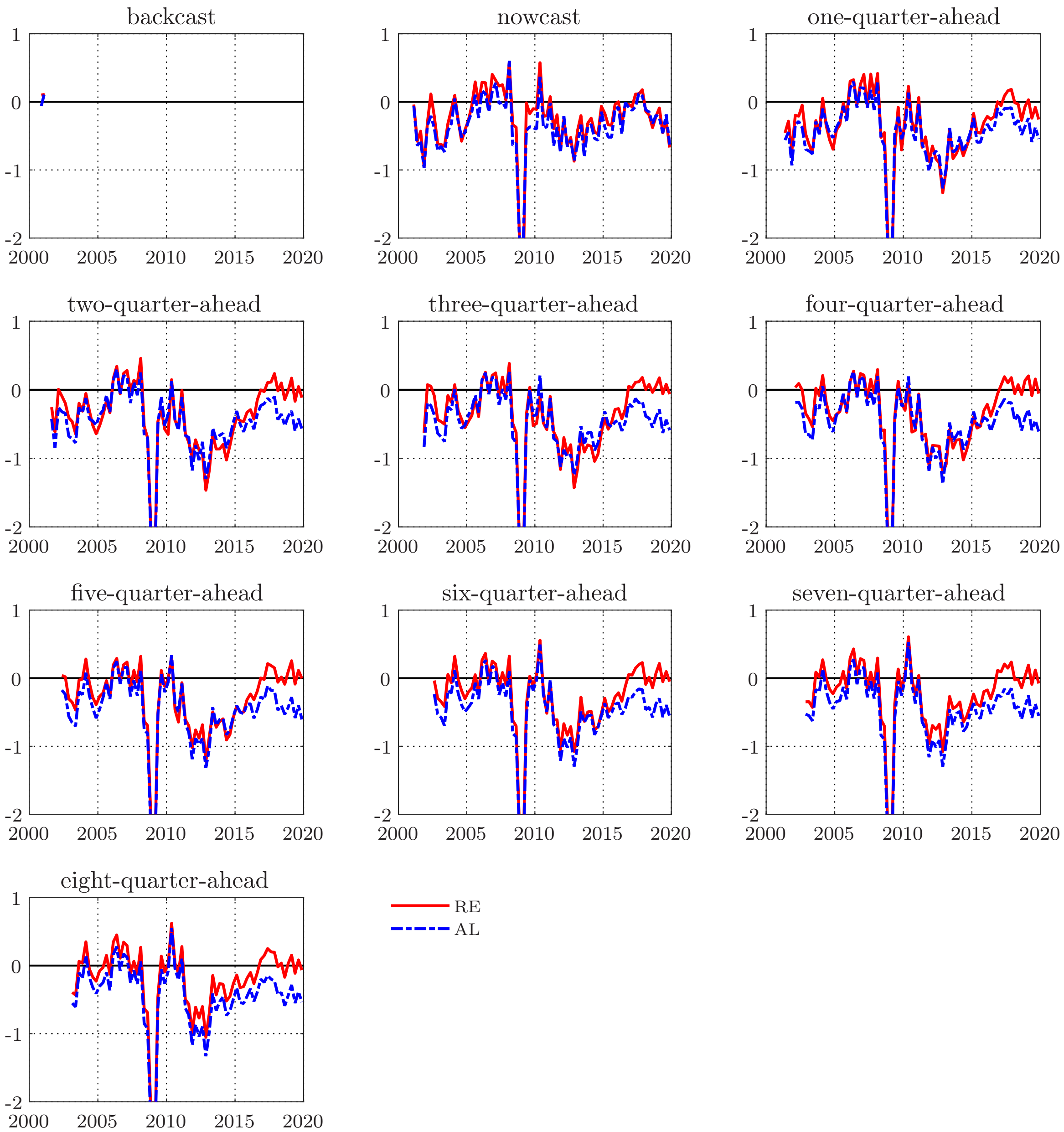


Figure C.3: Prediction errors of GDP deflator inflation covering the vintages 2001Q1–2019Q4.

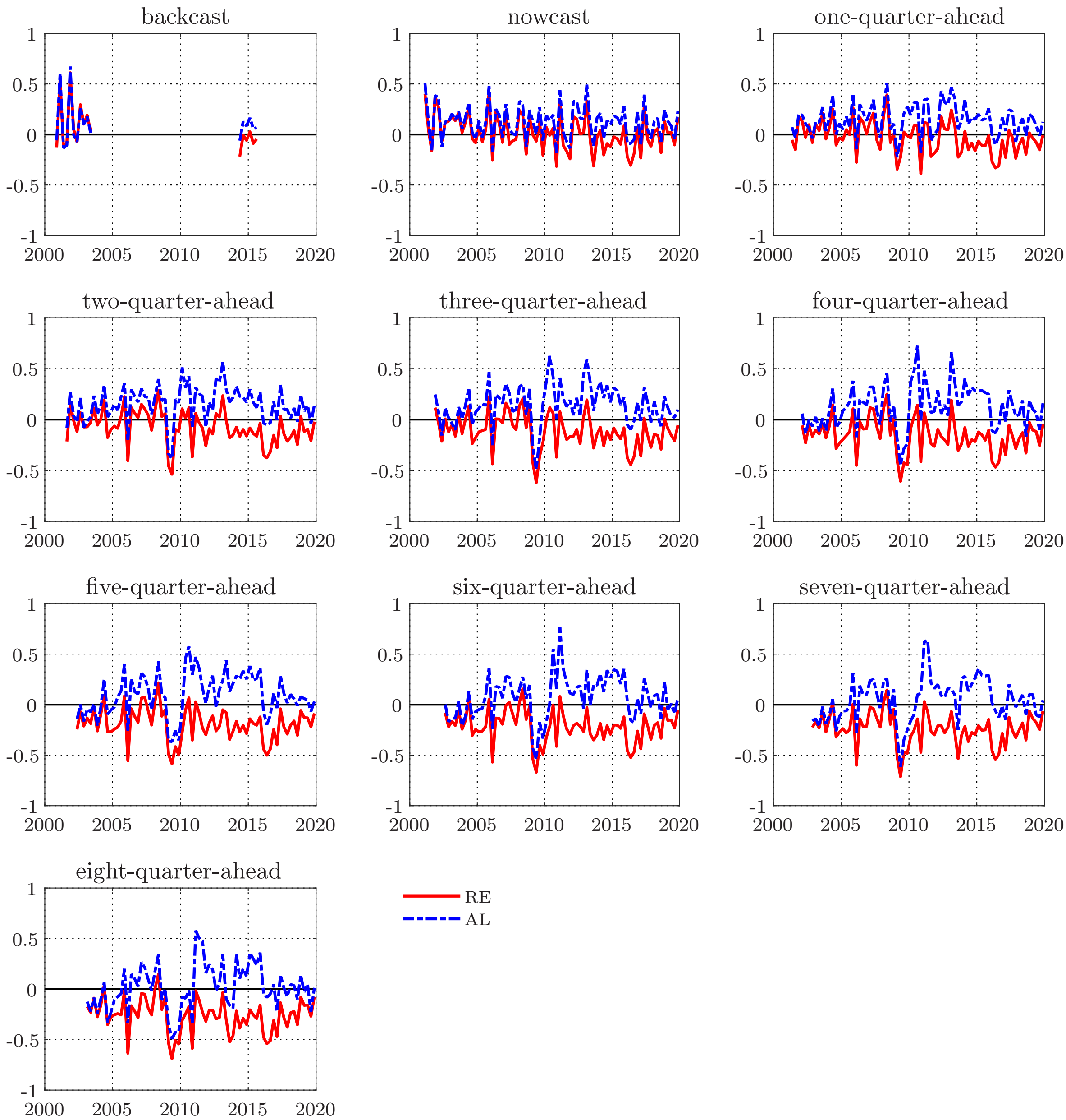


Figure C.4: Recursive estimates of the average log scores of the real GDP growth density forecasts covering the vintages 2001Q1–2019Q4.

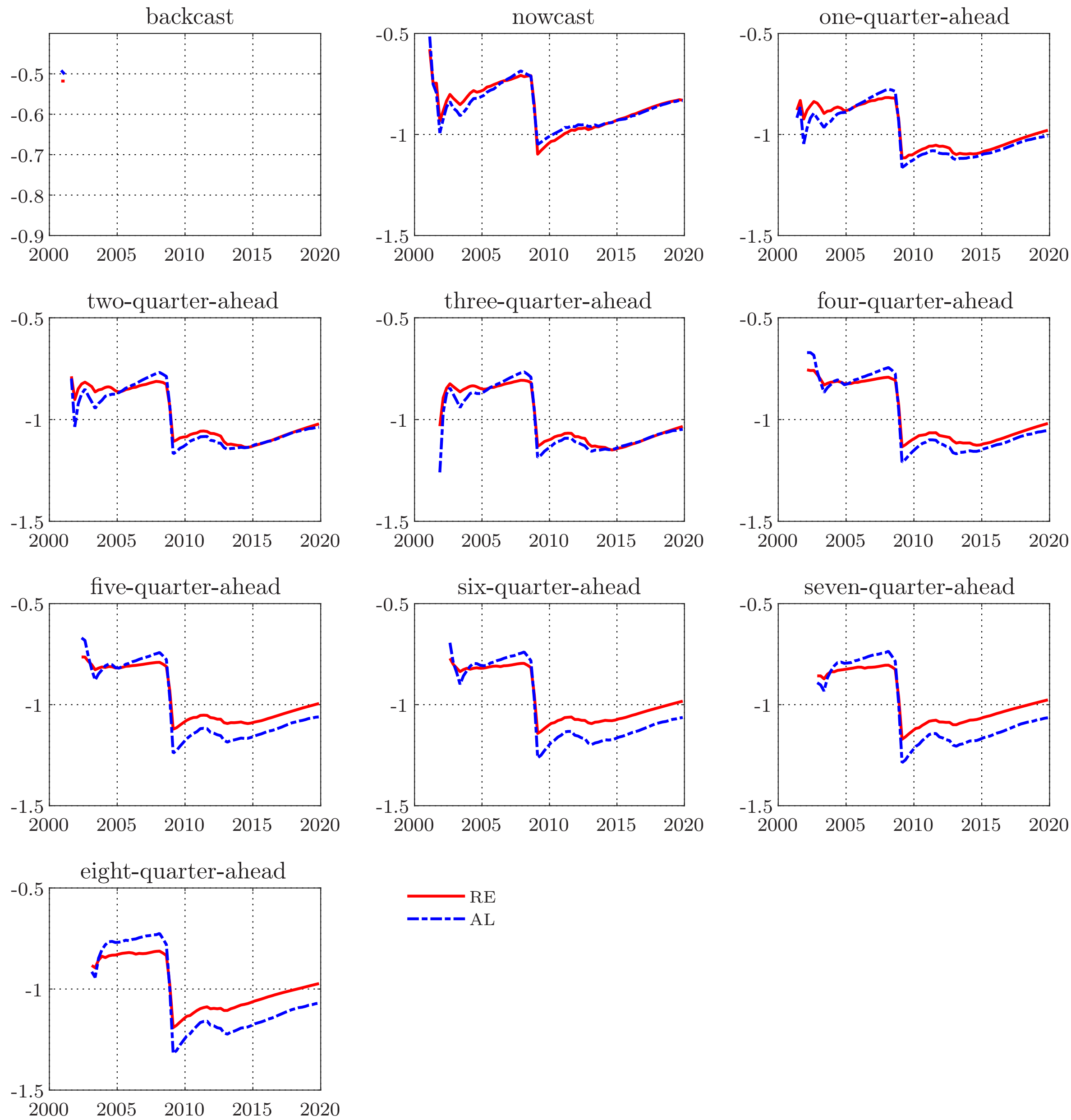


Figure C.5: Recursive estimates of the average log scores of the inflation density forecasts covering the vintages 2001Q1–2019Q4.

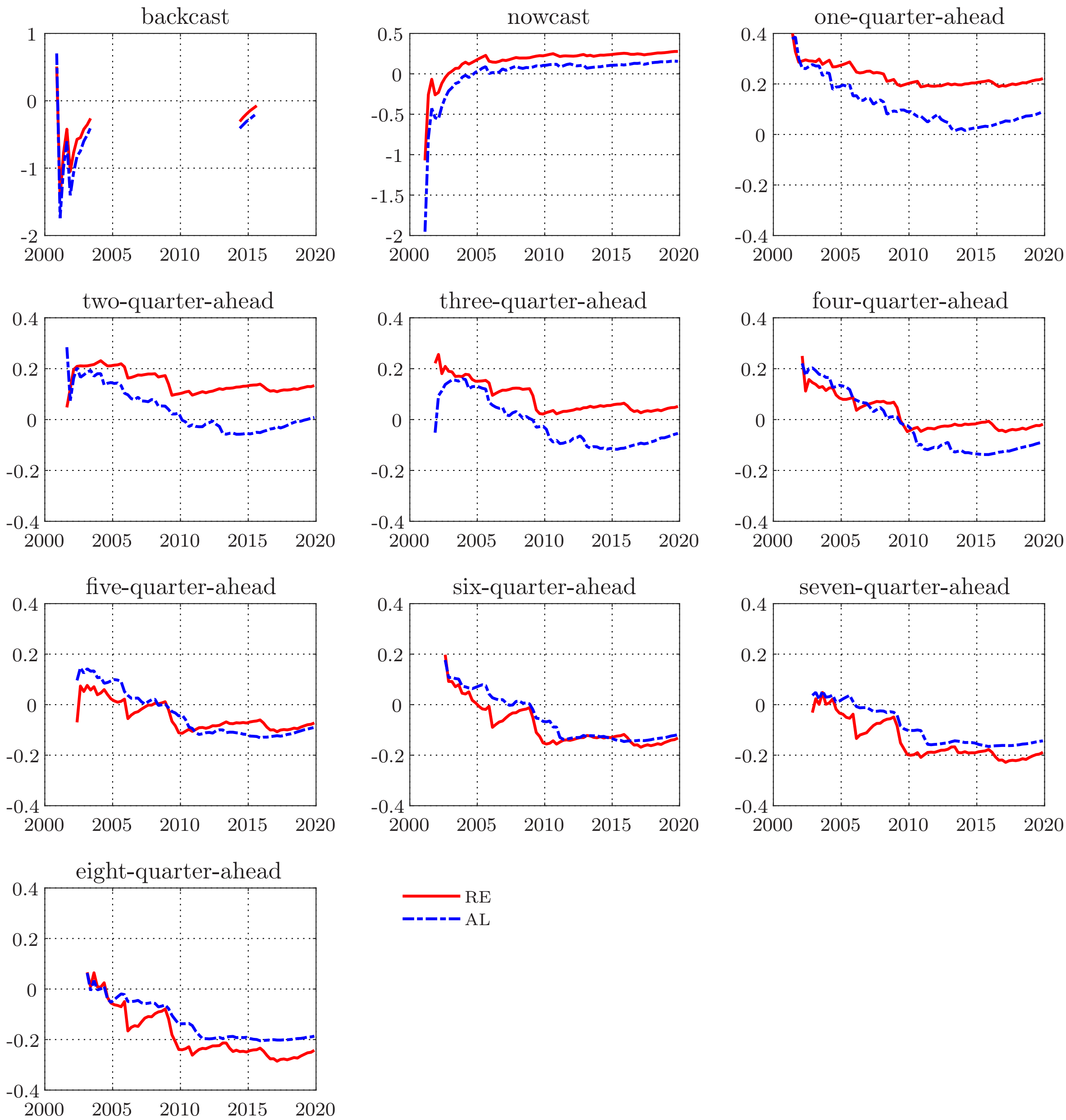


FIGURE C.6: Recursive estimates of the average continuous ranked probability scores of the real GDP growth distributional forecasts covering the vintages 2001Q1–2019Q4.

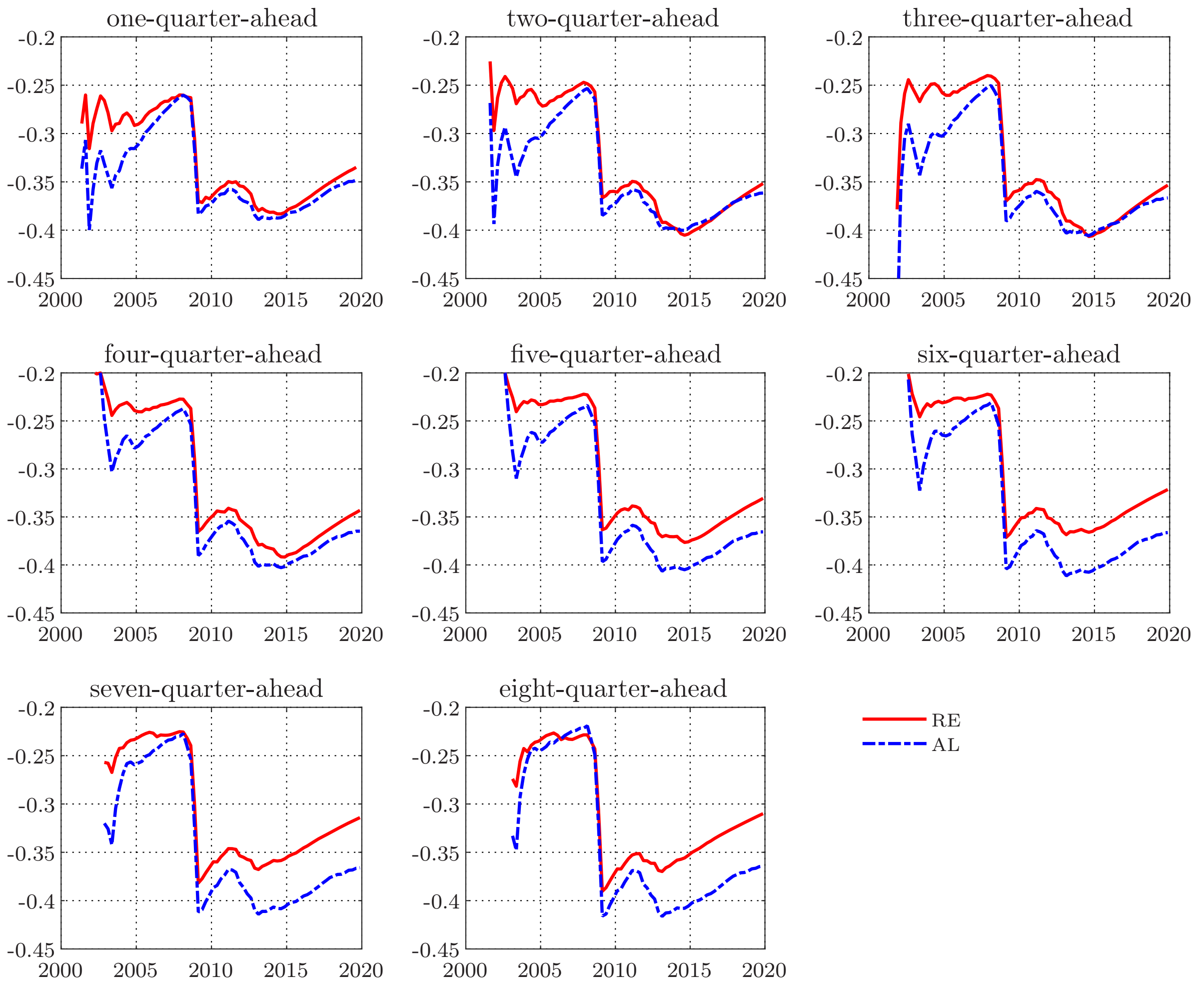


Figure C.7: Recursive estimates of the average continuous ranked probability scores of the inflation distributional forecasts covering the vintages 2001Q1–2019Q4.

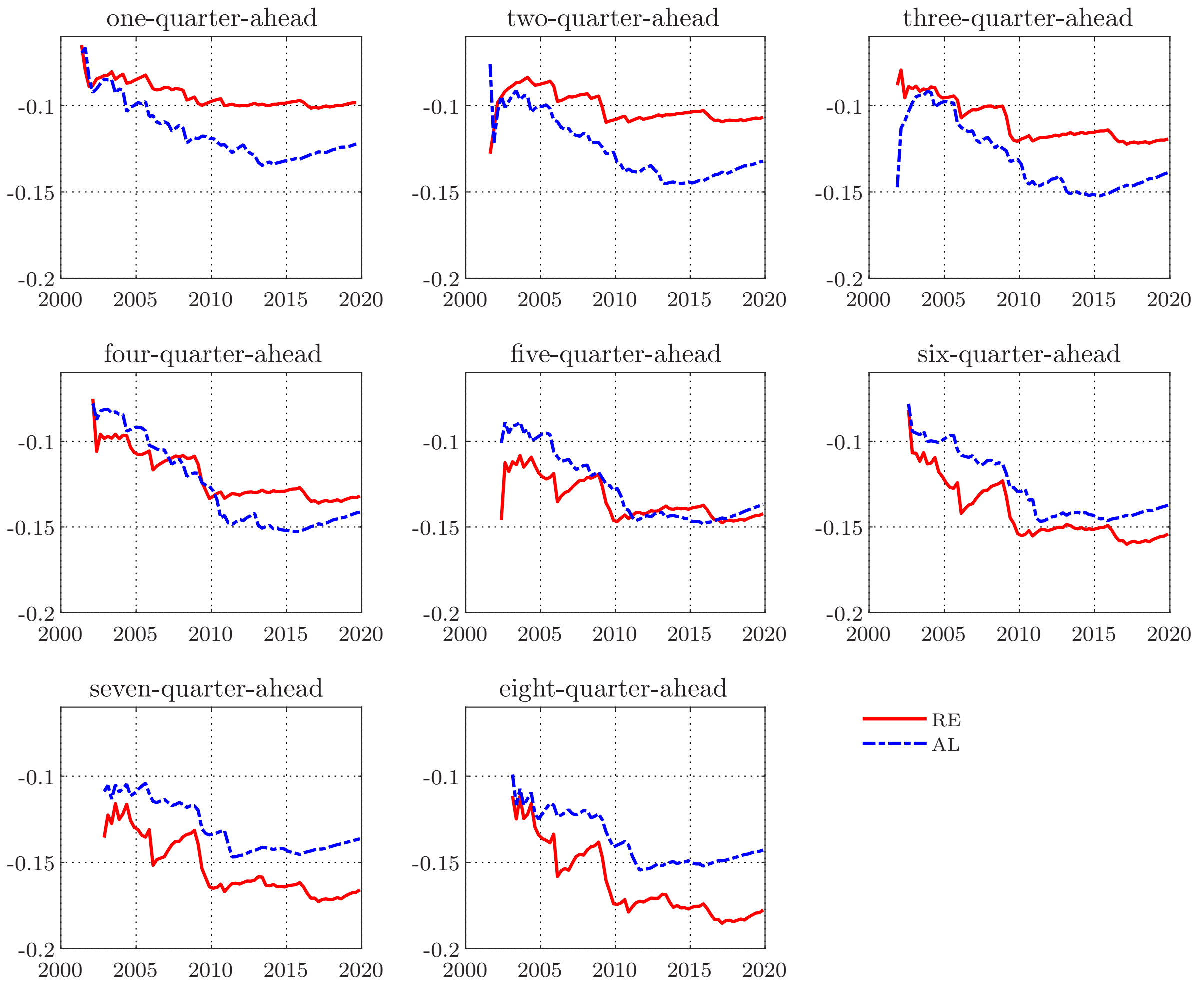
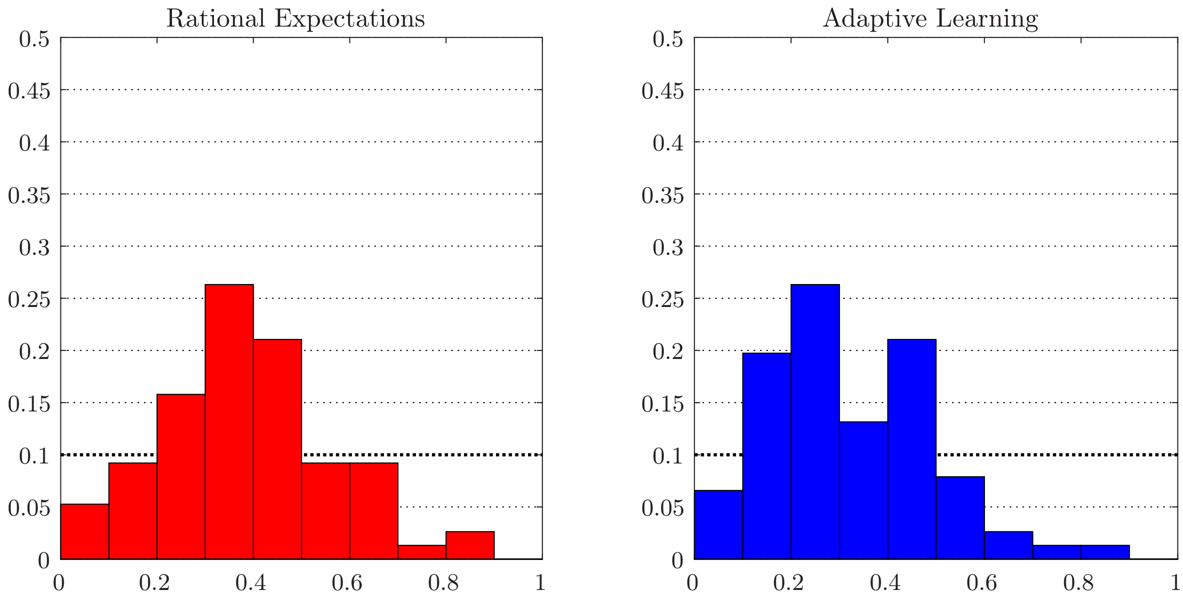
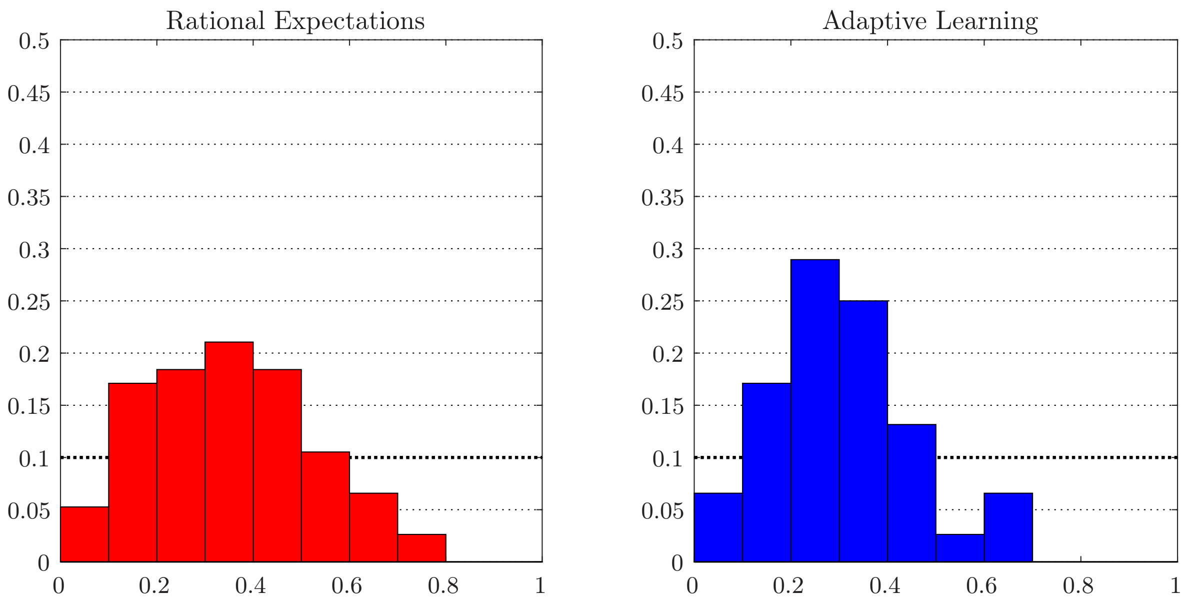


FIGURE C.8: Histograms of the estimated $\pi_{T+h|T}$ values for the marginal real GDP growth density forecasts at the nowcast ($h = 0$) and one-quarter-ahead ($h = 1$) horizons for 2001Q1–2019Q4.

A. Nowcasts



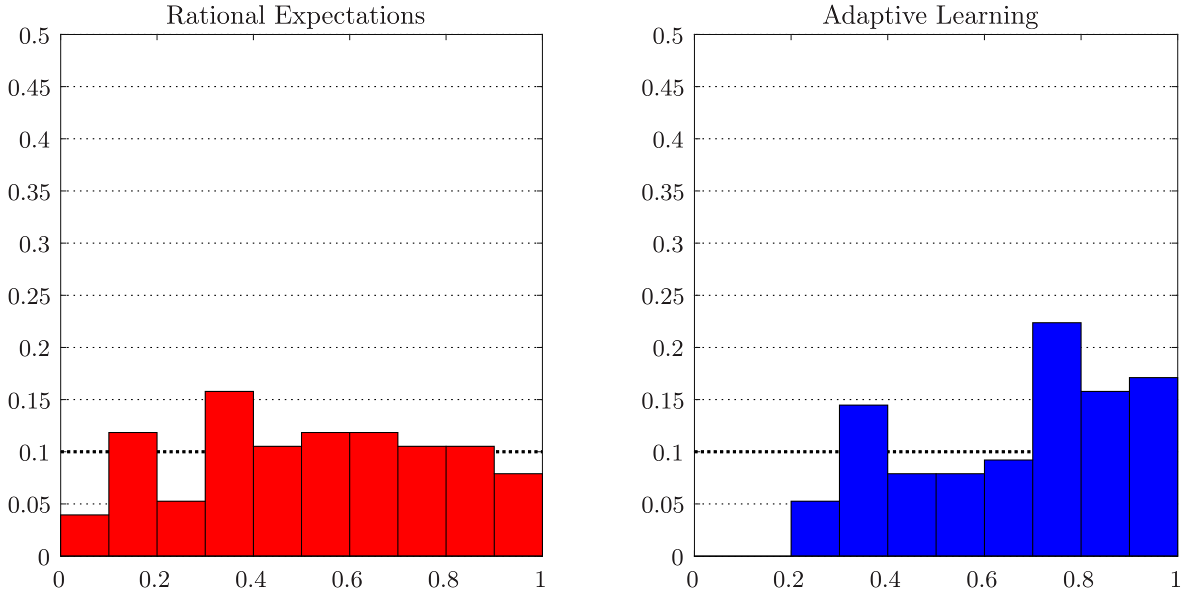
B. One-quarter-ahead forecasts



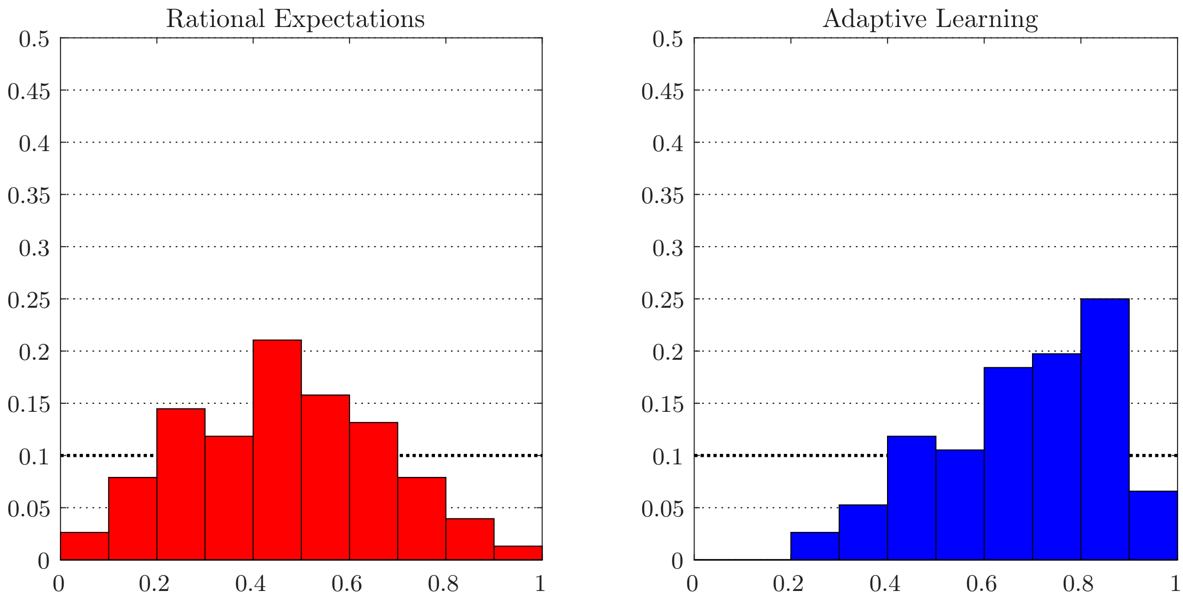
NOTES: The horizontal axis shows the 10 bins while the vertical axis shows the occurence frequency for the estimated π's. If these variables are uniformly distributed for a model, then the occurence in large samples is 0.10 for all bins.

FIGURE C.9: Histograms of the estimated $\pi_{T+h|T}$ values for the marginal GDP deflator inflation density forecasts at the nowcast ($h = 0$) and one-quarter-ahead ($h = 1$) horizons for 2001Q1–2019Q4.

A. Nowcasts



B. One-quarter-ahead forecasts



NOTES: See Figure C.8.